Consensus in a heterogeneous influence network

Wen Yang, Lang Cao, Xiaofan Wang,* and Xiang Li

Department of Automation, Shanghai Jiao Tong University, Shanghai, 200240, People's Republic of China

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We consider a dynamical network model in which a number of agents all move on the plane with the same constant absolute velocity. At each time step, each agent's direction is updated as the average of its direction plus the directions of other agents who can influence it. The influencing capability of each agent is represented by its influencing radius, which is randomly chosen according to a power-law distribution with a scaling exponent between 2 and ∞. As the value of the scaling exponent decreases, the radius distribution becomes more heterogeneous and the network becomes much easier to achieve direction consensus among agents due to the leading roles played by a few hub agents. Furthermore, almost all agents will finally move in the same desired direction in a strong heterogeneous influence network, if and only if a small fraction of hub agents can be controlled to move in the desired direction. These results also reflect the "robust yet fragile" feature of a heterogeneous influence network.

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Recent years have witnessed an increasing interest in the distributed coordination of multiple autonomous agents. In particular, a lot of efforts have been directed toward trying to understand how a network of mobile agents such as crowds of people, flocks of birds, schools of fish, or teams of robots, can cluster in formation without centralized coordination [1,2]. Most of these researches assumed that each agent has the same sensing capability. A classical example of such completely homogeneous sensing models [3] was proposed by Vicsek *et al.* [1]. In the Vicsek model, a number of agents move on the plane with the same absolute velocity but different directions. Each agent's direction is updated according to a local rule based on the average of its own direction plus the directions of its neighbors. Neighbors of an agent are those agents sensed by the agent and they are within a closed disk of the same sensing radius centered at the agent's current position [1]. Vicsek *et al.* provided a variety of simulation results to demonstrate that all agents may eventually move in the same direction, despite the absence of centralized coordination and the fact that each agent's set of nearest neighbors changes with time as the system evolves.

A completely homogeneous sensing model can equivalently be described as a completely homogeneous influencing model in which each agent adjusts its behavior based on the behaviors of those agents who can influence it [3]. However, in many social networks, the distributions of the influencing capabilities of agents are heterogeneous. A typical example is consensus decision making among a group of people. In the group, each person tries to persuade other people to accept her or his opinion and at the same time try to adjust her or his opinion based on the opinions of those people who can influence him. Clearly, different people may have different influence capabilities. A powerful leader may influence many people, while a little boy may influence only a few. Therefore, it is of practical importance to investigate collective behaviors in heterogeneous influencing networks.

Recently, the ability to synchronize in a complex dynamical network with respect to some kind of heterogeneity of the network (such as heterogeneous distribution of degree or intensity [4–6], parameter dispersion [7]) has been investigated. A basic assumption in these researches is that the topology of the network is fixed and a common result is that synchronizability is suppressed as the network becomes more heterogeneous [4,5,7], with an exception that better synchronizability for the Watts-Strogatz small world network is induced as the heterogeneity of the degree distribution is increased [6].

In this work, we investigate the collective behavior in a heterogeneous influencing network model which is a modification of the homogeneous sensing Vicsek model. The influencing capability of an agent is represented by its influencing radius, which is randomly chosen according to power-law distribution $P(r) \sim r^{-\gamma}$ with a scaling exponent $\gamma \in [2,\infty]$. As the value of the scaling exponent decreases, the radius distribution becomes more heterogeneous. The aim of this work is to investigate the ability of the network to achieve global direction consensus among agents in the sense that most agents will eventually move in the same direction in terms of the influence heterogeneity of the network. The main conclusion is that agreement among a few powerful leaders in a heterogeneous influence network is the key to achieving global direction consensus for the whole system.

As in the Vicsek model [1], we consider n autonomous agents, labeled 1 through n, all moving in a square-shaped cell of linear size L with periodic boundary conditions. Each agent has the same absolute velocity v but with different directions θ, which are randomly distributed in $[0,2\pi)$. Originally, agents are randomly distributed in the cell. In our model, we assume that agent i transmits its direction information to any other agents within a closed influencing disk of radius r_i centered at agent i. Each agent's direction is updated simultaneously based on the average of its own direction plus the directions of those agents whose influencing disks contain it. Mathematically, agent i's direction θ_i evolves in discrete time in accordance with a model of the form

*Corresponding author; Email address: xfwang@sjtu.edu.cn

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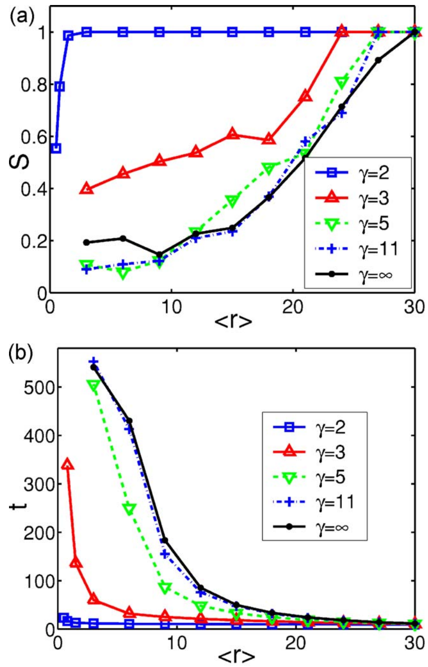

FIG. 1. (Color online) The relative size S of the largest cluster (a) and the convergence time (b) as functions of the average influencing radius $\langle r\rangle$ with various scaling exponents γ. System parameters are taken as $L=50$, $n=1250$, and $v=0.1$. All quantities are averaged over 30 realizations.

$$\theta_i(t+1) = \frac{1}{1+n_i(t)}\left(\theta_i(t) + \sum_{j\in N_i(t)} \theta_j(t)\right), \tag{1}$$

where $n_i(t)$ and $N_i(t)$ represent the number and set of agents who can influence agent i at time step t, respectively. The position of agent i is updated according to [1]

$$\mathbf{x}_i(t+1) = \mathbf{x}_i(t) + \mathbf{v}_i(t)\Delta t, \tag{2}$$

where the velocity vector $\mathbf{v}_i(t)$ of agent i has an absolute value v and a direction given by the angle $\theta_i(t)$.

We further assume that each agent's influencing radius is chosen randomly according to the power-law distribution $P(r)\sim r^{-\gamma}$ with a scaling exponent $\gamma\in[2,\infty]$ [8]. When $\gamma=\infty$, the distribution of influencing radii is completely homogeneous, which implies that all agents have the same influencing radius. In this case, our model is equivalent to the Vicsek model. As the value of the exponent γ decreases, the network becomes more heterogeneous in the sense that fewer agents have evidently larger influencing radii, whereas more agents have the relatively smaller influencing radii.

To evaluate the degree of direction consensus among agents, we denote a cluster as a group of agents with the same direction. We are interested in the relative size S of the largest cluster, which is defined as the ratio of the number of agents within the largest cluster to the total number of agents of the network when the model evolves to a steady state, which means that all agents' directions remain invariant. Note that $0\leqslant S\leqslant 1$, and a high value of S implies the high degree of direction consensus. If $S=1$, complete direction consensus is achieved that all agents move in one direction.

To concentrate on the influence of heterogeneity represented by the scaling exponent parameter γ, we fix system parameters n, L, and v [9]. Suppose that there are $n=1250$ agents moving in a square-shaped cell of linear size $L=50$ at the same absolute velocity $v=0.1$. Figure 1(a) shows the relative size S of the largest cluster as a function of the average influencing radius $\langle r\rangle$ with different values of the scaling exponent γ. For a fixed exponent γ, we can see that S is an increasing function of $\langle r\rangle$, due to the fact that increasing $\langle r\rangle$ improves the connectivity of the network. Furthermore, for any value of γ, there exists a threshold $\bar{r}(\gamma)$ so that complete consensus can be achieved (i.e., $S=1$) if $\langle r\rangle \geqslant \bar{r}(\gamma)$. We can see clearly that $\bar{r}(\gamma)$ is an increasing function of γ. In particular, in the most homogeneous case (i.e., $\gamma=\infty$), $\bar{r}(\infty)\approx 30$; while in the most heterogeneous case (i.e., $\gamma=2$), $\bar{r}(2)\approx 3$, which is just about one tenth of $\bar{r}(\infty)$. This implies that it is much easier to achieve complete consensus as the network becomes more heterogeneous. Furthermore, for any given value of $\langle r\rangle$, S is a decreasing function of γ, implying that more agents will finally move in the same direction as the network becomes more heterogeneous.

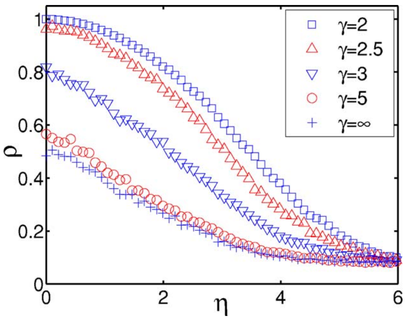

FIG. 2. (Color online) The order parameter ρ as a function of the noise amplitude η with various exponents γ. Other system parameters are taken as $\langle r\rangle=3$, $n=100$, $L=31$, and $v=1$. All estimates are the results of averaging over 100 realizations.

We also compute the convergence time defined as the time required to arrive at the steady state. As shown in Fig. 1(b), the convergence time decreases when γ decreases or $\langle r\rangle$ increases. This implies that the required time to achieve consensus can also be shortened as the network becomes more heterogeneous or has better connectivity.

The heuristic reason for these phenomena lies in the fact that most agents in a strong heterogeneous network have very small influencing radii, but a few hub agents have very large influencing radii. In the ideal case, we can approximately assume that these hub agents form a complete subgraph, which can influence almost all other agents but cannot be influenced by those agents. Since a sufficient condition for consensus in a network of agents is that the network keeps strongly connected during the evolution process [10], those hub agents will eventually move in the same direction, denoted as $\bar{\theta}$. Suppose that the number of hub agents is n_h, and each other agent i is influenced by at least one of the hub agents. Under these ideal assumptions, we have

$$\theta_i(t+1) = \{1/[1+n_i(t)]\}[\theta_i(t) + n_i(t)\bar{\theta}], \tag{3}$$

where $1\leqslant n_i(t)\leqslant n_h$, which leads to

$$\theta_i(t+1) - \bar{\theta} = \{1/[1+n_i(t)]\}[\theta_i(t) - \bar{\theta}] \leq \frac{1}{2}[\theta_i(t) - \bar{\theta}]. \quad (4)$$

This implies that the direction of any other agents will also tend to $\bar{\theta}$.

The simulation results and heuristic analysis suggest that a few hub agents in a heterogeneous influence network play the leading roles during the system evolution process, and agreement among the hub agents is the key to achieving the nearly complete consensus in the whole network.

To investigate the influence of noise on our model, we rewrite the direction update equation (1) as follows:

$$\theta_i(t+1) = \frac{1}{1+n_i(t)}\left(\theta_i(t) + \sum_{j \in N_i(t)} \theta_j(t)\right) + \Delta\theta_i(t), \quad (5)$$

where $\Delta\theta_i(t)$ is a random noise chosen with a uniform probability from the interval $[-\eta/2, \eta/2]$, $\eta \geq 0$. In this case, we cannot partition all the agents into different clusters accurately. Therefore, we adopt the order parameter ρ to evaluate the consensus degree, which is defined as [1]

$$\rho = \frac{1}{nv}\left|\sum_{i=1}^{n} \mathbf{v}_i\right|. \quad (6)$$

Clearly, as shown in Fig. 2, ρ is a decreasing function of η, which implies that the consensus degree decreases as the noise amplitude increases. On the other hand, for any fixed value of η, ρ is a decreasing function of γ, which means that as the network becomes more heterogeneous, it is more robust to noise disturbance.

Now suppose that we not only want to achieve the direction consensus but also want all agents to move in a given desired direction. Clearly, to achieve this goal, some kinds of control strategies need to be applied to the system. The leading role of the hub agents in an influence network motivates us to explore the possibility of achieving global consensus in a desired direction just by controlling a few leading agents.

Pinning control has been a common technique for the control of spatiotemporal chaos in regular dynamical networks and has recently been applied to scale-free dynamical networks [11–13]. Suppose that we want to achieve consensus in an influence network with a desired direction θ_p by pinning a fraction of agents. We compare two pinning control schemes. In the specifically pinning scheme, a fraction f of agents with the largest radii are pinned, while in the randomly pinning scheme, a fraction f of randomly selected agents are pinned. The pinned agents move at the same absolute velocity as other agents but with the fixed and desired direction θ_p. Other agents update their directions according to Eq. (1). To evaluate the degree of consensus in the desired direction, we define the relative size S_d of the largest desired cluster as the ratio of the largest number of agents moving in the desired direction θ_p to the total number of agents of the network when the system evolves to a steady state.

We compare the effects of specifically and randomly pinning control schemes on an influence network. The average influencing radius is taken as $\langle r \rangle = 6$. Figure 3 shows the relative size S_d of the largest desired cluster as a function of the fraction f of pinned agents with different scaling exponents γ. For each pinning control scheme and each value of γ, there exists a critical value of f above which all the agents will move in the desired direction (i.e., $S_d = 1$). In a highly homogeneous influence network with a relatively large scaling exponent (e.g., $\gamma = 5$), the difference between the critical values of two pinning control schemes is relatively small [see Fig. 3(c)]. This is due to the fact that almost all the agents in the network have similar influencing abilities. On the other hand, there is a sharp difference between two critical values in a highly heterogeneous network. For example, in the case of $\gamma = 2$, the critical value corresponding to the randomly pinning control scheme is 0.43, which is about 10 times larger than the critical value 0.04 of the specifically pinning control scheme [see Fig. 3(a)]. The origin of this sharp difference still lies in the heterogeneity feature: As explained before, if those hub agents with large radii move in the desired direction, then almost all other agents will also move in the desired direction. On the other hand, since most agents have small radii, the probability is very high that the small fraction of randomly chosen agents all have small ra-

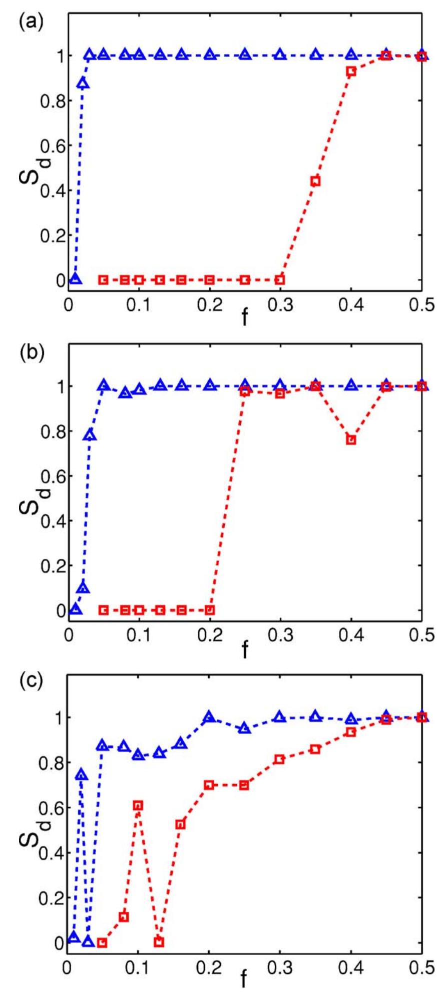

FIG. 3. (Color online) The relative size S_d of the largest desired cluster when a fraction f of randomly (□) or specifically (△) chosen agents are pinned at the desired direction. (a) $\gamma = 2$, (b) $\gamma = 3$, (c) $\gamma = 5$. Other system parameters are taken as $\langle r \rangle = 6$, $n = 1250$, $L = 50$, and $v = 0.1$. All estimates are the results of averaging over 30 realizations.

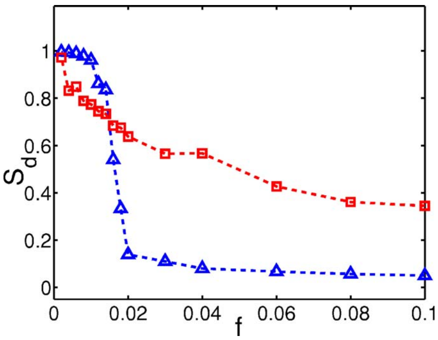

FIG. 4. (Color online) The relative size S of the largest cluster when the directions of a fraction f of randomly (□) or specifically (△) selected agents are randomly chosen and fixed through the evolution. System parameters are taken as $\langle r\rangle=6$, $n=1250$, $L=50$, and $v=0.1$. All estimates are the results of averaging over 30 realizations.

dii, and pinning control of these agents cannot influence most of the other agents.

In the pinning control schemes, we assume that the directions of a small fraction of selected agents are fixed at the same direction. Now suppose that the directions of a small fraction of selected agents are fixed at different directions, can consensus still be achieved among other agents in the network? To investigate this question, we specifically or randomly choose a fraction f of agents. Directions of these agents are randomly chosen and fixed through the evolution process. Other agents update their directions according to Eq. (1). As shown in Fig. 4, in a high heterogeneous network with $\gamma=2$, the relative size S of the largest cluster corresponding to the specific scheme decreases rapidly from $S=1$ as f increases: the largest consensus cluster contains only about 10% of the agents in the network if the directions of only 2% of the agents with the largest influencing radii are fixed at different directions. This implies that agreement among hub agents is also a necessary condition for the emergence of global consensus. On the other hand, the relative size S of the largest cluster corresponding to the random scheme decreases very slowly from $S=1$ as f increases: the largest consensus cluster contains nearly 40% of the agents in the network even if the directions of 10% of the random selected agents are fixed at different directions. This result demonstrates that the ability to achieve global consensus in a heterogeneous influence network is robust to random errors but fragile to specific attacks. Disagreement among a small number of specifically chosen hub agents with large radii can significantly destroy the emergence of global consensus, while disagreement among a small fraction of random selected agents with small radii cannot have a significant influence on consensus among other agents in the system. This conclusion is consistent with the recent discovery that the connectivity of a heterogeneous network is error tolerant but vulnerable to attacks [14].

In conclusion, numerical simulations and heuristic analysis indicate that the ability to achieve direction consensus in an influence network is enhanced as the heterogeneity of the influencing radius distribution increases. In particular, global consensus with a desired direction can be achieved in a heterogeneous influence network only if a small number of leading agents can be controlled to move along the desired direction. These results may shed some light on achieving a desired consensus in social networks and other man-made multiagent systems.

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